

WILDER MANIFOLDS ARE LOCALLY ORIENTABLE

BY GLEN E. BREDON

RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY

Communicated by R. L. Wilder, May 26, 1969

Abstract.—A proof is given for the long-standing conjecture of R. L. Wilder that every generalized manifold is locally orientable. Roughly speaking, a generalized n -manifold is a locally compact space whose local homology groups at each point are those of an n -manifold. Local orientability is a condition in which the local homology groups at neighboring points have a certain nice relationship to one another. Local orientability is indispensable for almost all applications.

R. L. Wilder conjectured,¹ more than 20 years ago, that every generalized manifold is locally orientable. The conjecture is, without doubt, the main unsolved problem concerning generalized manifolds. We shall present a surprisingly simple proof of it in this note.

We shall make use of the Borel-Moore homology theory and sheaf cohomology, and our notation will be that of reference 2. (In particular, H_* denotes homology with arbitrary closed supports, while H_*^c denotes that with compact supports. The former is analogous to homology with locally finite chains, and the latter is analogous to ordinary homology.) The base ring L will be assumed to be a principal ideal domain.

Definition: A locally compact space M will be called a Wilder n -manifold (over L) if it is *clc*, of finite cohomology dimension² over L , the homology sheaf² $\mathcal{H}_i(M; L) = 0$ for $i \neq n$, and the "orientation sheaf" $\mathcal{O} = \mathcal{H}_n(M; L)$ has each stalk a free L -module of rank one. M is said to be locally orientable if \mathcal{O} is locally constant.

Remarks: Our definition agrees with that of a *clc* $L - n$ space in the terminology of Borel³ (but *not* in that of ref. 2). By Borel,³ p. 26, this agrees with what Borel calls a Wilder n -manifold when L is a field, and this agrees with the original notion of Wilder¹ (who considers only the field case). See also Raymond.⁴ Locally orientable Wilder n -manifolds coincide with n -*cms* in the notation of references 2 and 3 (see ref. 2, p. 241). The condition *clc* is stronger than we need and it would suffice to assume only local connectivity of M for the proof of the theorem below.

THEOREM. *Every Wilder n -manifold M is locally orientable.*

The following lemma gives the main part of the proof.

LEMMA. *If M is a connected, paracompact, Wilder n -manifold over a field K and if σ is a nontrivial global section of \mathcal{O} , then σ is nonzero at every point of M .*

Proof: Let the closed set A be the support of σ . Suppose that $A \neq M$. Since K is a field, σ induces a homomorphism from the constant sheaf K into \mathcal{O} which is an isomorphism over A . Thus

$$H_c^n(A; K) \approx H_c^n(A; \mathcal{O}|_A) \approx H_0^c(M, M - A; K) = 0,$$

by Poincaré duality (ref. 2, pp. 209–210) and by the exact sequence

$$H_0^c(M - A; K) \twoheadrightarrow H_0^c(M; K) \rightarrow H_0^c(M, M - A; K) \rightarrow 0.$$

The exact sequence

$$H_c^n(M - A; K) \xrightarrow{j^*} H_c^n(M; K) \rightarrow H_c^n(A; K) = 0$$

shows that j^* is onto. Now $H_c^n(M - A; K)$ can be identified with $\text{dir. lim } H_c^n(U; K)$ where U ranges over the open, paracompact sets with compact closure in $M - A$. (This is to avoid a paracompactness assumption on $M - A$.) By Kronecker duality (ref. 2, p. 184)

$$j_*: H_n(M; K) \rightarrow \text{inv. lim. } H_n(U; K)$$

is a *monomorphism*. However, by Poincaré duality, j_* can be identified with section restriction:

$$\Gamma(\mathcal{O}) = H^0(M; \mathcal{O}) \rightarrow \text{inv. lim } H^0(U; \mathcal{O}|U) = \text{inv. lim } \Gamma(\mathcal{O}|U) = \Gamma(\mathcal{O}|M - A).$$

Since $\sigma_*\Gamma(\mathcal{O})$ restricts to zero on $M - A$, this is a contradiction.

Proof of the theorem: We remark that it follows from the universal coefficient theorem (ref. 2, p. 186) that if p is a prime in L and $K = L/pL$, then M is a Wilder n -manifold over K with orientation sheaf $\mathcal{O} \otimes K$. (Change of rings must be used here; ref. 2, p. 238.) Let x be a point of M and let σ be a local section of \mathcal{O} which gives a generator of the stalk at x . Since x has a neighborhood basis consisting of paracompact open sets, we may assume that σ is defined on a connected, paracompact, open neighborhood U of x . σ induces a homomorphism of the constant sheaf L on U to $\mathcal{O}|U$ which is an isomorphism on the stalk over x . By the lemma, the induced map $L/pL \rightarrow (\mathcal{O} \otimes L/pL)|U$ is an *isomorphism* for each prime p in L . Now a homomorphism $L \rightarrow L$ of modules, which induces an isomorphism $L/pL \rightarrow L/pL$ for every prime p of L , is clearly an isomorphism itself. Thus the homomorphism $L \rightarrow \mathcal{O}|U$ of sheaves over U is an isomorphism on each stalk, and hence is an isomorphism. Thus \mathcal{O} is constant over U .

It follows easily from this theorem, and various known or easily provable facts, that all reasonable definitions of (co-)homology manifolds are equivalent, at least if one assumes *clc* and coefficients in the integers or a field. We illustrate this vague statement by observing the following simple cohomological criterion:

COROLLARY. *If the base ring L is the integers or a field, then a locally compact space M is an n -cm over L (refs. 2, 3) if and only if M is clc_L , of finite cohomological dimension over L , and $H^*(M, M - \{x\}; L) \approx H^*(D^n, S^{n-1}; L)$ for all points x in M .*

Proof: Note that $H^i(M, M - \{x\}; L) \approx \text{dir. lim. } \tilde{H}^{i-1}(U - \{x\}; L)$ over neighborhoods U of x . We also remark that this result (modulo local orientability, of course) is due to Raymond⁴ in case L is a field.

Note that $H_i^c(M, M - \{x\}; L)$ is the stalk $\mathcal{H}_i(M; L)_x$ of the homology sheaf at x (ref. 2, page 206). From the universal coefficient sequence for *clc* pairs (ref. 2, p. 232) we obtain the exact sequence

$$0 \rightarrow \text{Ext}(\mathcal{H}_{i-1}(M; L)_x; L) \rightarrow H^i(M, M - \{x\}; L) \rightarrow \text{Hom}(\mathcal{H}_i(M; L)_x; L) \rightarrow 0.$$

It follows that $H^*(M, M - \{x\}; L)$ is of finite type iff $H_*(M; L)_x$ is. (For the case $L = \mathbb{Z}$ see ref. 2, p. 236.) Thus the exact sequence shows that the condition in the corollary is equivalent to $\mathcal{H}_*(M; L)_x \approx H_*(D^n, S^{n-1}; L)$ for all x ; that is, to M being a Wilder n -manifold. Thus the corollary follows from the theorem and the known fact that n -cms coincide with locally orientable Wilder n -manifolds (ref. 2, p. 241).

¹ Wilder, R. L., "Topology of Manifolds," *Am. Math. Soc. Colloq. Publ.*, vol. 32 (1949).

² Bredon, G. E., *Sheaf Theory* (New York: McGraw-Hill, 1967).

³ Borel, A., *et al.*, "Seminar on Transformation Groups," *Annals of Mathematics Studies*, No. 46 (Princeton, N.J.: Princeton University Press, 1960),

⁴ Raymond, F., "Local cohomology groups with closed supports," *Math. Z.*, **76**, 31-41 (1961)